



12.6 Finding the eigenvalues of a matrix

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Introduction

- In this topic, we will
 - See various techniques for finding eigenvalues
 - Look at some examples
 - Introduce the QR decomposition





Eigenvalues of a matrix

- We will discuss algorithms for finding the eigenvalues of a matrix
 - We'll start with 2×2 matrices
 - We'll investigate 3×3 matrices
 - We'll look at the general case of $n \times n$ matrices





Eigenvalues of a 2×2 matrix

- For a 2×2 matrix:

$$\begin{aligned}\det(\lambda I_2 - A) &= \det \begin{pmatrix} \lambda - a_{1,1} & -a_{1,2} \\ -a_{2,1} & \lambda - a_{2,2} \end{pmatrix} = (\lambda - a_{1,1})(\lambda - a_{2,2}) - a_{1,2}a_{2,1} \\ &= \lambda^2 - (a_{1,1} + a_{2,2})\lambda + a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \\ &= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)\end{aligned}$$

$$\begin{aligned}\frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4 \det(A)}}{2} &= \frac{\operatorname{tr}(A)}{2} \pm \sqrt{\left(\frac{\operatorname{tr}(A)}{2}\right)^2 - \det(A)} \\ &= \frac{\operatorname{tr}}{2} \pm \sqrt{\left(\frac{\operatorname{tr}}{2}\right)^2 - \det}\end{aligned}$$





Eigenvalues of a 2×2 matrix

- For example, consider

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$$

$$\text{tr}(A) = 10 \quad \therefore \frac{\text{tr}(A)}{2} = 5$$

$$\det(A) = 1$$

$$\begin{aligned} \lambda_1, \lambda_2 &= 5 \pm \sqrt{25 - 1} \\ &= 5 \pm 2\sqrt{6} \end{aligned}$$





Eigenvalues of a 2×2 matrix

- For example, consider

$$A = \begin{pmatrix} -2.9 & -2.7 \\ 2.4 & 0.7 \end{pmatrix}$$

$$\text{tr}(A) = -2.2 \quad \therefore \frac{\text{tr}(A)}{2} = -1.1$$

$$\det(A) = 4.45$$

$$\lambda_1, \lambda_2 = -1.1 \pm \sqrt{1.21 - 4.45}$$

$$= -1.1 \pm \sqrt{-3.24}$$

$$= -1.1 \pm 1.8j$$





Eigenvalues of a 3×3 matrix

- For finding the eigenvalues of a 3×3 matrix:

$$\det(\lambda I_3 - A) = \det \begin{pmatrix} \lambda - a_{1,1} & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & \lambda - a_{2,2} & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & \lambda - a_{3,3} \end{pmatrix} = (\lambda - a_{1,1})(\lambda - a_{2,2})(\lambda - a_{3,3})$$
$$-a_{1,2}a_{2,3}a_{3,1} - a_{3,2}a_{2,1}a_{1,3}$$
$$-(\lambda - a_{1,1})a_{2,3}a_{3,2}$$
$$-(\lambda - a_{3,3})a_{1,2}a_{2,1}$$
$$-(\lambda - a_{2,2})a_{1,3}a_{3,1}$$

$$\begin{pmatrix} \lambda - a_{1,1} & -a_{1,2} & -a_{1,3} & | & \lambda - a_{1,1} & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & \lambda - a_{2,2} & -a_{2,3} & | & -a_{2,1} & \lambda - a_{2,2} & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & \lambda - a_{3,3} & | & -a_{3,1} & -a_{3,2} & \lambda - a_{3,3} \end{pmatrix}$$





Eigenvalues of a 3×3 matrix

- Unless you can easily detect an eigenvalue,
you must solve the cubic, which means using software
 - If the matrix is not invertible,
you are guaranteed 0 is an eigenvalue, so 0 is a root

$$\lambda^3 + c_2\lambda^2 + c_1\lambda = 0$$

$$\begin{aligned}\lambda_1, \lambda_2, \lambda_3 &= 0, \frac{-c_2 \pm \sqrt{c_2^2 - 4c_1}}{2} \\ &= 0, -\frac{c_2}{2} \pm \sqrt{\left(\frac{c_2}{2}\right)^2 - c_1}\end{aligned}$$

$$\lambda^3 + c_2\lambda^2 = 0$$

$$\lambda_1, \lambda_2, \lambda_3 = 0, 0, -c_2$$

$$\lambda^3 = 0 \quad \lambda_1, \lambda_2, \lambda_3 = 0, 0, 0$$





Eigenvalues of a 3×3 matrix

- Now, if we are calculating the eigenvalues of a real matrix

$$\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0$$

- By the fundamental theorem of algebra,
one root must be real
- If you can detect that root, or find it using Newton's method,
you can factor that root out using polynomial long division

$$\begin{array}{r} \lambda^2 + \tilde{c}_1\lambda + \tilde{c}_0 \\ \lambda - \lambda_1 \overline{) \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0} \end{array}$$

$$\lambda_1, \lambda_2, \lambda_3 = \lambda_1, \frac{-\tilde{c}_1 \pm \sqrt{\tilde{c}_1^2 - 4\tilde{c}_0}}{2}$$





Eigenvalues of a 3×3 matrix

- For example, consider

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 5 & 3 \\ 2 & 3 & 5 \end{pmatrix}$$

- The characteristic polynomial is:

$$\lambda^3 - 14\lambda^2 + 48\lambda$$

- We note that $\lambda_1 = 0$ is a root, and therefore an eigenvalue
- The roots of $\lambda^2 - 14\lambda + 48$ may be found with the quadratic formula:

$$\frac{14 \pm \sqrt{196 - 4 \cdot 48}}{2} = 7 \pm \frac{\sqrt{4}}{2} = 7 \pm 1$$

- Thus, the eigenvalues are 0, 6, 8





Eigenvalues of a 3×3 matrix

- For example, consider

$$A = \begin{pmatrix} 4 & -2 & -3 \\ -2 & 5 & 2 \\ -3 & 2 & 4 \end{pmatrix}$$

$$\begin{array}{r} \lambda^2 - 12\lambda + 27 \\ \lambda - 1 \overline{) \lambda^3 - 13\lambda^2 + 39\lambda - 27} \\ \underline{-(\lambda^3 - \lambda^2)} \\ -12\lambda^2 + 39\lambda \\ \underline{-(-12\lambda^2 + 12\lambda)} \\ 27\lambda - 27 \\ \underline{-(27\lambda - 27)} \\ 0 \end{array}$$

- The characteristic polynomial is:

$$\lambda^3 - 13\lambda^2 + 39\lambda - 27$$

- We note that $\lambda_1 = 1$ is a root, and therefore an eigenvalue

- The roots of $\lambda^2 - 12\lambda + 27$ may be found with the quadratic formula:

$$\frac{12 \pm \sqrt{144 - 4 \cdot 27}}{2} = 6 \pm \frac{\sqrt{36}}{2} = 6 \pm 3$$

- Thus, the eigenvalues are 1, 3, 9





Eigenvalues of a $n \times n$ matrix

- For larger matrices, the techniques are based on the following:

$$A_0 \leftarrow A$$

- A QR decomposition decomposes a matrix into:

- An isometric (orthogonal) matrix Q
- An upper-triangular (“right-triangular”) matrix R

$$A_0 = Q_0 R_0$$

$$A_1 \leftarrow R_0 Q_0$$

- We repeat this many times

$$A_k = Q_k R_k$$

$$A_{k+1} \leftarrow R_k Q_k$$

- The diagonal entries of $Q_k R_k$ converge to the eigenvalues





Eigenvalues of a $n \times n$ matrix

- For example, consider this matrix:

$$A = \begin{pmatrix} 0.706934 & 0.072311 & 0.985703 & 0.340629 \\ 0.861327 & 0.180285 & 0.622810 & 0.027637 \\ 0.678343 & 0.260757 & 0.486343 & 0.166809 \\ 0.707802 & 0.936198 & 0.186391 & 0.460085 \end{pmatrix}$$

- It has eigenvalues

$$1.8942, -0.0005391 + 0.3377j, -0.0005391 - 0.3377j, -0.059433$$

$$Q_{37}R_{37} = \begin{pmatrix} 1.8942 & -0.057262 & -0.75208 & -0.28950 \\ 0.3 \times 10^{-28} & -0.13602 & -0.33912 & -0.0056144 \\ 0.8 \times 10^{-27} & 0.39046 & 0.13495 & 0.78977 \\ 0.9 \times 10^{-55} & 0.3 \times 10^{-28} & 0.6 \times 10^{-28} & -0.059433 \end{pmatrix}$$





Summary

- Following this topic, you now
 - Know how to find eigenvalues of 2×2 or 3×3 matrices
 - Understand there are algorithms for finding eigenvalues of larger matrices
 - Have had a brief introduction to the QR decomposition





References

- [1] <https://en.wikipedia.org/wiki/Eigenvalues>





Acknowledgments

None so far.





Colophon

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